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# Some properties of time-machines

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**Abstract.** Closed time-like loops on space–time manifolds are discussed with a view to finding out whether time-machines exist. It is shown that a three-surface associated with causality violation, the 'time-machine boundary', has topology $T^2 \times R^1$. This torus begins its existence as a singular closed null geodesic ($C_1$), in agreement with a result due to Tipler. Null curves close to $C_1$ carry off angular momentum to future null infinity, and it is speculated that this radiation process damps the rotation of a system to below the critical angular momentum at which causality violation occurs.

## 1. Introduction

One of the worrying aspects of relativity is that many of the exact solutions to the field equations admit of closed time-like loops. These world lines would represent the path of an object which is able to meet its previous and future self. In particular, the rotating black hole Kerr solution (Carter 1968) and the vacuum exterior of an infinitely long rotating cylinder (Tipler 1974) both violate causality in this way. This suggests that a finite rotating object, if spun sufficiently rapidly, would act as a time-machine.

It is generally agreed that physical space–time should not violate causality, and so the question we are asking is: 'How far is relativity prepared to go in dismissing time-machines?' The answer should be regarded as a reflection upon the powers and weaknesses of the methods used in relativity, not as a statement as to whether time-machines exist—physics must already assume that they do not.

Tipler (1976) has shown that a singularity will generically be associated with time-machines, provided we are not inside a black hole. He therefore invokes the 'cosmic censorship' hypothesis and suggests that causality violation will be restricted to the interior of an event horizon. This hope might seem to be supported by the fact that time-like loops on the Kerr solution are indeed inside the hole under reasonable conditions.

However, the displeasing nature of time-machines is independent of their location, and so the 'cosmic censorship' hypothesis is not sufficiently powerful to solve the problem; what we require is a simple causal explanation as to why time-machines will not come to exist. In this paper we show that a rotating object will radiate an unlimited amount of angular momentum to future null infinity when causality violation is imminent, and this is interpreted as a simple (albeit imprecise) reason for the non-existence of time-machines.

  

## 2. Time-machine boundaries

We can isolate the causality violating parts of a manifold with a causality horizon, so that all closed time-like curves are wholly contained within this horizon. A more interesting three-surface, it turns out, is the 'time-machine boundary', which has the defining property that it encloses the smallest region of the manifold in which every loop is at least partly contained.

Typically this is equivalent to a more useful definition: The time-machine boundary is that closed three-surface which divides the manifold into an interior and an exterior so that the exterior is the largest submanifold on which the stable causality condition holds.' (See Hawking and Ellis 1973, p 198, to be referred to as HE).

The best example of a time-machine boundary is provided by the Kerr metric, on which every loop is at least partly contained within the surface defined by $g\hat{\phi}\hat{\phi}=0$, where $\hat{\phi}$ is a killing azimuthal angle coordinate in the Boyer–Lindquist system (Carter 1968). This surface ($r_{\mathrm{T}}$) is a torus for each value of the time-coordinate ($\hat{t}$) and so in this case the time-machine boundary has topology $T^2\times R^1$. This proves to be true generally.

*Theorem.* A compact time-machine boundary has topology $T^2\times R^1$ (where $R^1$ represents a line-segment).

*Proof.* The stable causality condition holds on the exterior submanifold, hence, by HE, § 6.4.9, p 198, there is a function $f$ on the exterior whose gradient is time-like. Since the exterior is the largest such submanifold then the gradient of a suitably differentiable $f$ is null on the time-machine boundary.

The perpendiculars to the gradient of $f$ lie in the surface $f=$ constant, and as we have a Lorentz signature on $M$ then there is one and only one null vector in $f=$ constant at the time-machine boundary. We show that this vector ($\boldsymbol{c}$) is tangent to the time-machine boundary. The intersection of the time-machine boundary and a three-surface $f=$ constant is a closed compact two-surface ($T$). Set up local pseudo-orthonormal coordinates ($\delta x^{\mu}$) at some point P in $T$, so that $\partial/\partial x^1$ and $\partial/\partial x^2$ are tangent to $T$, $\partial/\partial x^3$ is perpendicular to $T$ in $f=$ constant and $\partial/\partial x^4$ is parallel to the gradient of $f$. Suppose $\boldsymbol{c}$ is not tangent to $T$, then $\partial/\partial x^1$ and $\partial/\partial x^2$ must be space-like, and there is a combination $A\partial/\partial x^3+B\partial/\partial x^4$ which is also space-like. The volume element described by these three space-like vectors provides a way of extending $f=$ constant while keeping the gradient of $f$ time-like. This is not possible by HE, § 6.4.9, and so $\boldsymbol{c}$ is tangent to $T$.

At each point of $T$ there is a null vector tangent to $T$, and so there is a continuously defined vector field on $T$. This implies that the Euler characteristic of $T$ is zero and so $T$ is either a torus or a Klein bottle. Since $M$ is both space and time orientable $T$ is a torus.

By taking $T$ for each value of $f$, we arrive at the topology $T^2\times R^1$, the result being independent of the topology of $M$.

## 3. The torus

A torus has the property that two independent vector fields may be defined on it, whereas we have used only one. Thus we can associate a winding number $W$ with the null curve on $T$, which gives the number of twists the curve makes in moving once around $T$. $W$ is a topological quantity which changes as $f$ increases, so that typically $W$

is irrational and we have the pleasing (but useless) result that the typical torus is generated by one null curve. This curve will not in general be geodesic.

If the torus does not exist for all time (as it does in the Kerr case) then it commences its existence on some three-slice $f = f_0$. On $f_0$ $T$ will be a circle ($C_1$) generated by one closed null curve. (This is only so if $M$ is sufficiently devoid of symmetries, but this is no restriction for what follows.) Similarly the torus ends its existence as a null curve ($C_2$) on some $f$ greater than $f_0$. We are concerned with the curve $C_1$.

*Theorem.* $C_1$ is a null geodesic. (Similarly for $C_2$.)

*Proof.* Suppose $C_1$ is not geodesic, then, by HE, § 4.5.10, p 112, there is a time-like curve between any two points of $C_1$. This curve intersects $f = f_0$ both towards the future and to the past and so a continuous choice of the orientation of the future null cone cannot be made on $f_0$. $C_1$ is geodesic.

*Theorem.* If the weak energy condition and the generic condition hold on $C_1$, then $C_1$ is incomplete.

*Proof.* If the conditions hold and $C_1$ is complete then use HE, §§ 4.4.5 and 4.5.12 to show that there would be a time-like curve between two points of $C_1$. This is impossible, again, so $C_1$ is incomplete.

This is an extension of Tipler's result that there is a singularity associated with time-machines—the extension being that the result is true inside black holes. We might therefore conclude that in a collapse to a Kerr solution the torus $r_T$ breaks off from the ring singularity—that is, that the singularity is formed before the manifold becomes causality violating. This, I think, would be incorrect; we wish to be rid of time-machines not to provide a description of how they come about in 'realistic' situations. Instead we ask what type of singularity makes $C_1$ incomplete. There are two possibilities—a scalar polynomial singularity or a parallelly propagated singularity.

As discussed by Hawking and Ellis (HE, p 190 and chapter 8) a scalar polynomial singularity is a point removed from $M$ where some scalar measure of curvature diverges, and a parallelly propagated singularity arises in connection with a closed null geodesic. If we parallelly propagate the tangent vector to $C_1$ $(\partial/\partial V)$ around $C_1$ it becomes $(\partial/\partial V)'$ so that

$$(\partial/\partial V)' = a(\partial/\partial V). \tag{1}$$

This has consequences for the affine distance on $C_1$, so that $C_1$ is future incomplete if $a > 1$, and past incomplete if $a < 1$ being complete only if $a = 1$.

I shall take the view that the singularity on $C_1$ is a parallelly propagated singularity, for the following reasons. Primarily, there is no good reason why there should exist a scalar polynomial singularity on $C_1$; there is no obvious cause for a divergent curvature scalar, whereas the black hole singularity theorems, for example, *do* suggest intense fields in that there exists a closed trapped surface on $M$. Thus it would seem to demand chance to get a scalar polynomial singularity on $C_1$.

Secondly, examine the factor $a$. From the geodesic equation

$$\frac{\partial u^\alpha}{\partial V} = -\Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma \tag{2}$$

we may redefine $a$;

Let $\partial/\partial x^4$ be parallel to $\partial/\partial V$ on $C_1$, $\partial/\partial x^3$ is the perpendicular future null vector (the gradient of $f$) and $\partial/\partial x^1$ and $\partial/\partial x^2$ are perpendicular unit space-like vectors.

$$\Delta u^4 = -\oint_{C_1} \Gamma^4_{44}\, u^4\, \mathrm{d}x^4 \qquad \text{and} \qquad \Delta u^4 = (a-1)u^4 \tag{3}$$

$\Delta u^1$, $\Delta u^2$ and $\Delta u^3$ are, of course, zero, but imposing this restriction does not restrain $\Delta u^4$, so that typically $a$ will not be unity; that is, we already expect the typical closed null geodesic to be incomplete without invoking scalar polynomial singularities—the set of complete closed null geodesics is of measure zero on the set of all closed null geodesics.

Accepting that $C_1$ is parallely propagated incomplete the problem is to find the value of $a$ and the consequences of this value.

Using the above definition of $a$ we see that a curve which is arbitrarily close to $C_1$ (for all values of its affine parameter in the incomplete direction) will also suffer from the same incompleteness—the factor $a$ is a property of the local space-time near $C_1$, not just a property of $C_1$. Thus, a null geodesic generator of $\dot{J}^+(C_1)$ will be past incomplete if $a < 1$. Such geodesics are, in fact, those considered by Tipler, and he showed that they are indeed past incomplete, suggesting that $a$ on $C_1$ is less than one. However, a result of Hawking and Ellis (HE § 6.4.4.) shows that if $C_1$ exists then it must have $a$ greater than one; ('If $C$ is a future incomplete closed null geodesic then there is a variation of this curve to the future which yields a closed time-like curve.')

The view here is that $C_1$ does not exist and therefore $\dot{J}^+(C_1)$ does not exist and so we must compute $a$ via equation (3).

The metric near $C_1$ is approximately

$$\mathrm{d}s^2 = -2\mathrm{d}x^3\, \mathrm{d}x^4 + \mathrm{d}x^{12} + \mathrm{d}x^{22} \tag{4}$$

*independent* of the existence of $C_1$, so that

$$\Delta u^4 \simeq -\oint_{(\text{near } C_1)} \tfrac{1}{2} g_{44,3} u^4\, \mathrm{d}x^4 \tag{5}$$

since derivatives in direction $\partial/\partial x^4$ are roughly zero. $g_{44,3}$ is less than zero near $C_1$ and so $a$ is greater than one.

## 4. That $C_1$ should not exist

Suppose that the geometry has developed up to some $f$ just less than $f_0$, so that there is a closed space-like curve ($C_s$) arbitrarily close to some fictitious $C_1$. We want to know whether the geometry will develop so that $C_1$ will come to exist. There are null curves close to $C_s$ which are not closed—they violate the strong causality condition, but not the stable causality condition. These null curves can be treated as neighbours of the hypothetical $C_1$, and we can use the equation of geodesic deviation to find out if they remain close to $C_1$, and thus suffer from the same future incompleteness as $C_1$, or whether they diverge from $C_1$.

$b^\alpha$ is the separation vector between $C_1$ and a nearby (and existent) null curve. At some point $\partial b^\alpha/\partial V$ is zero.

$$\frac{\partial^2 b^\alpha}{\partial V^2} = -R^\alpha{}_{\beta\gamma\delta} b^\gamma u^\beta u^\delta.$$

Integrating twice;

$$Db^{\alpha} = -\iint R^{\alpha}{}_{\beta\gamma\delta} b^{\gamma} \, dx^{\beta} \, dx^{\delta}$$

where the integration is taken around $C_1$ an infinite number of times. To a first approximation the quantity $b^{\gamma}$ can be considered a constant so that, to the next approximation, $Db^{\alpha}$ diverges. That is, the null curves initially close to and roughly parallel to $C_s$ leave the locality of $C_s$ after a *finite* affine distance—they do not suffer from the same future incompleteness as $C_1$.

Consider the momentum ($\boldsymbol{p}$) of a photon on one of these null curves;

$$\boldsymbol{p} = \frac{\partial}{\partial V} = \boldsymbol{u}. \tag{6}$$

So long as the curve stays near $C_1$, then $\boldsymbol{u}' = a\boldsymbol{u}$ and $\boldsymbol{p}' = a\boldsymbol{p}$ which shows that the momentum of a photon can become arbitrarily large before it radiates away to future null infinity. Thus, some measure of the angular momentum of the photon also becomes unbounded, and this angular momentum is also radiated to future null infinity.

Conservation theorems tell us that the asymptotic measure of angular momentum is constant, and so the angular momentum of the rotating system can only decrease.

## 5. Conclusion

Objects which are rotating sufficiently rapidly to be on the verge of violating causality tend to radiate angular momentum in unlimited quantities. This is perhaps the reason why time-machines do not come to exist.

More complete calculations could certainly be done using particular models of time-machines, although it might prove more difficult to arrive at general theorems about the immediate effects of the radiation.

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